

# Explicit quasi-periodic wave solutions and asymptotic analysis to the supersymmetric Ito's equation

Engui Fan<sup>a,\*</sup>, Y. C. Hon<sup>b</sup>,

a. School of Mathematics Sciences, Fudan University, Shanghai 200433, PR China

b. Department of Mathematics, City University of Hong Kong, Hong Kong, PR China

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**Abstract:** Based on a Riemann theta function and the super-Hirota bilinear form, we propose a key formula for explicitly constructing quasi-periodic wave solutions of the supersymmetric Ito's equation in superspace  $\mathbb{C}_\Lambda^{2,1}$ . Once a nonlinear equation is written in bilinear forms, then the quasi-periodic wave solutions can be directly obtained from our formula. The relations between the periodic wave solutions and the well-known soliton solutions are rigorously established. It is shown that the quasi-periodic wave solutions tends to the soliton solutions under small amplitude limits.

## 1. Introduction

The Ito's equation takes the form

$$u_{tt} + 6(u_x u_t)_x + u_{xxx} = 0, \quad (1.1)$$

which was first proposed by Ito, and its bilinear Bäcklund transformation, Lax representation and multi-soliton solutions were obtained [1]. The other integrable properties of this equation such as the nonlinear superposition formula, Kac-Moody algebra, bi-Hamiltonian structure have been further found [2]-[5]. Recently, Liu, Hu and Liu proposed the following supersymmetric Ito's equation [6]

$$\mathfrak{D}_t F_t + 6(F_x(\mathfrak{D}_t F))_x + \mathfrak{D}_t F_{xxx} = 0, \quad (1.2)$$

and obtained its one-, two- and three-soliton solutions, where  $F = F(x, t, \theta)$  is fermionic superfield depending on usual even independent variable  $x, t$  and odd Grassman variable  $\theta$ . The differential operator  $\mathfrak{D}_t = \partial_\theta + \theta \partial_t$  is the super derivative.

The bilinear derivative method developed by Hirota is a powerful approach for constructing exact solution of nonlinear equations[7]–[13]. Based on the Hirota bilinear form and the Riemann theta functions, Nakamura presented an approach to directly construct a kind of quasi-periodic solutions of nonlinear equation [14, 15], where the periodic wave solutions of the KdV equation

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\*Electronic mail: faneg@fudan.edu.cn.

and the Boussinesq equation were obtained. This method not only conveniently obtains periodic solutions of a nonlinear equation, but also directly gives the explicit relations among frequencies, wave-numbers, phase shifts and amplitudes of the wave. Recently, this method is further developed to investigate the discrete Toda lattice, (2+1)-dimensional Kadomtsev-Petviashvili equation and Bogoyavlenskii's breaking soliton equation[16]-[20] .

Our present paper will considerably improve the key steps of the above method so as to make the method much more lucid and straightforward for applying a class of nonlinear supersymmetric equations. First, the above method will be generalized into the supersymmetric context. The quasi-periodic solutions of supersymmetric equations still seem not investigated to our knowledge. Second, we a formula that the Riemann theta functions satisfy a super-Hirota bilinear equation. This formula actually provides us an uniform method which can be used to construct quasi-periodic wave solutions of nonlinear differential, difference and supersymmetric equations. Once a nonlinear equation is written in bilinear forms, then the quasi-periodic wave solutions of the nonlinear equation can be obtained directly by using the formula. As illustrative example, we shall construct quasi-periodic wave solutions to the supersymmetric Ito's equation (1.2). Moreover, we also establish the relations between our quasi-periodic wave solutions and the soliton solutions that were obtained by Liu and Hu [5] .

The organization of this paper is as follows. In section 2, we briefly introduce a super-Hirota bilinear that will be suitable for constructing quasi-periodic solutions of the equation (1.2). And then introduce a general Riemann theta function and provide a key formula for constructing periodic wave solutions. In section 3, as application of our formula, we construct one-periodic wave solutions to the equation (1.2). We further present a simple and effective limiting procedure to analyze asymptotic behavior of the one-periodic wave solutions. It is rigorously shown that the quasi-periodic wave solutions tends to the known soliton solutions obtained by Liu and Hu under "small amplitude" limits. At last, we briefly discuss the conditions on the construction of multi-periodic wave solutions of the equation (1.2) in section 4.

## 2. The superspace, Hirota bilinear form and the Riemann theta functions

To fix the notations and make our presentation self-contained, we briefly recall some properties about superanalysis and super-Hirota bilinear operators. The details about superanalysis refer, for instance, to Vladimirov's work [22, 23].

A linear space  $\Lambda$  is called  $Z_2$ -graded if it represented as a direct sum of two subspaces

$$\Lambda = \Lambda_0 \oplus \Lambda_1,$$

where elements of the spaces  $\Lambda_0$  and  $\Lambda_1$  are homogeneous. We assume that  $\Lambda_0$  is a subspace consisting of even elements and  $\Lambda_1$  is a subspace consisting of odd elements. For the element

$f \in \Lambda$  we denote by  $f_0$  and  $f_1$  its even and odd components. A parity function is introduced on the  $\Lambda$ , namely,

$$|f| = \begin{cases} 0, & \text{if } f \in \Lambda_0, \\ 1, & \text{if } f \in \Lambda_1. \end{cases}$$

We introduce an annihilator of the set of odd elements by setting

$${}^\perp \Lambda_1 = \{\lambda \in \Lambda : \lambda \Lambda_1 = 0\}.$$

A superalgebra is a  $Z_2$ -graded space  $\Lambda = \Lambda_0 \oplus \Lambda_1$  in which, besides usual operations of addition and multiplication by numbers, a product of elements is defined with the usual distribution law:

$$a(\alpha b + \beta c) = \alpha ab + \beta ac, \quad (\alpha b + \beta c)a = \alpha ba + \beta ca,$$

where  $a, b, c \in \Lambda$  and  $\alpha, \beta \in \mathbb{C}$ . Moreover, a structure on  $\Lambda$  is introduced of an associative algebra with a unite  $e$  and even multiplication i.e., the product of two even and two odd elements is an even element and the product of an even element by an odd one is an odd element:  $|ab| = |a| + |b| \pmod{2}$ .

A commutative superalgebra with unit  $e = 1$  is called a finite-dimensional Grassmann algebra if it contains a system of anticommuting generators  $\sigma_j, j = 1, \dots, n$  with the property:  $\sigma_j \sigma_k + \sigma_k \sigma_j = 0, j, k = 1, 2, \dots, n$ , in particular,  $\sigma_j^2 = 0$ . The Grassmann algebra will be denote by  $G_n = G_n(\sigma_1, \dots, \sigma_n)$ .

The monomials  $\{e_0, e_i = \sigma_{j_1} \cdots \sigma_{j_n}\}, j = (j_1 < \dots < j_n)$  form a basis in the Grassmann algebra  $G_n$ ,  $\dim G_n = 2^n$ . Then it follows that any element of  $G_n$  is a linear combination of monomials  $\sigma_{j_1} \cdots \sigma_{j_k}, j_1 < \dots < j_k$ , that is,

$$f = f_0 + \sum_{k \geq 0} \sum_{j_1 < \dots < j_k} f_{j_1 \dots j_k} \sigma_{j_1} \cdots \sigma_{j_k},$$

where the coefficients  $f_{j_1 \dots j_k} \in \mathbb{C}$ .

**Definition 1.** Let  $\Lambda = \Lambda_0 \oplus \Lambda_1$  be a commutative Banach superalgebra, then the Banach space

$$\mathbb{C}_\Lambda^{m,n} = \Lambda_0^m \times \Lambda_1^n$$

is called a superspace of dimension  $(m, n)$  over  $\Lambda$ . In particular, if  $\Lambda_0 = \mathbb{C}$  and  $\Lambda_1 = 0$ , then  $\mathbb{C}_\Lambda^{m,n} = \mathbb{C}^m$ .

A function  $f(\mathbf{x}) : \mathbb{C}_\Lambda^{m,n} \rightarrow \Lambda$  is said to be superdifferentiable at the point  $x \in \mathbb{C}_\Lambda^{m,n}$ , if there exist elements  $F_j(\mathbf{x})$  in  $\Lambda, j = 1, \dots, m+n$ , such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \sum_{j=1}^{m+n} \langle F_j(\mathbf{x}), h_j \rangle + o(\mathbf{x}, \mathbf{h}),$$

where  $\mathbf{x} = (x_1, \dots, x_m, x_{m+1}, \dots, x_n)$  with components  $x_j, j = 1, \dots, m$  being even variable and  $x_{m+j} = \theta_j, j = 1, \dots, n$  being Grassmann odd ones. The vector  $\mathbf{h} = (h_1, \dots, h_m,$

$h_{m+1}, \dots, h_{m+n})$  with  $(h_1, \dots, h_m) \in \Lambda_0^m$  and  $(h_{m+1}, \dots, h_{m+n}) \in \Lambda_1^n$ . Moreover,

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{\|o(\mathbf{x}, \mathbf{h})\|}{\|\mathbf{h}\|} \rightarrow 0.$$

The  $F_j(\mathbf{x})$  are called the super partial derivative of  $f$  with respect to  $x_j$  at the point  $\mathbf{x}$  and are denoted, respectively, by

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = F_j(\mathbf{x}), \quad j = 1, \dots, m+n.$$

The derivatives  $\frac{\partial f(\mathbf{x})}{\partial x_j}$  with respect to even variables  $x_j$ ,  $j = 1, 2, \dots, n$  are uniquely defined. While the derivatives  $\frac{\partial f(\mathbf{x})}{\partial \theta_j}$  to odd variables  $\theta_j = x_{j+n}$ ,  $j = 1, 2, \dots, m$  are not uniquely defined, but with an accuracy to within an addition constant  $c\sigma_1 \cdots \sigma_n$ ,  $c \in \mathbb{C}$  from an annihilator  ${}^\perp G_n$  of finite-dimensional Grassmann algebra  $G_n$ .

The super derivative also satisfies Leibniz formula

$$\frac{\partial(f(\mathbf{x})g(\mathbf{x}))}{\partial x_j} = \frac{\partial f(\mathbf{x})}{\partial x_j}g(\mathbf{x}) + (-1)^{|x_j||f|}f(\mathbf{x})\frac{\partial g(\mathbf{x})}{\partial x_j}, \quad j = 1, \dots, m+n. \quad (2.1)$$

Denote by  $\mathcal{P}(\Lambda_1^n, \Lambda)$  the set of polynomials defined on  $\Lambda_1^n$  with value in  $\Lambda$ . We say that a super integral is a map  $I : \mathcal{P}(\Lambda_1^n, \Lambda) \rightarrow \Lambda$  satisfying the following condition is an super integral about Grassmann variable

- (1) A linearity:  $I(\mu f + \nu g) = \mu I(f) + \nu I(g)$ ,  $\mu, \nu \in \Lambda$ ,  $f, g \in \mathcal{P}(\Lambda_1^n, \Lambda)$ ;
- (2) translation invariance:  $I(f_\xi) = I(f)$ , where  $f_\xi = f(\boldsymbol{\theta} + \boldsymbol{\xi})$  for all  $\boldsymbol{\xi} \in \Lambda_1^n$ ,  $f \in \mathcal{P}(\Lambda_1^n, \Lambda)$ .

We denote  $I(\theta^\varepsilon) = I_\varepsilon$ , where  $\varepsilon$  belongs to the set of multiindices  $N_n = \{\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n), \varepsilon_j = 0, 1, \boldsymbol{\theta}^\varepsilon = \theta_1^{\varepsilon_1} \cdots \theta_n^{\varepsilon_n} \neq 0\}$ . In the case when  $I_\varepsilon = 0$ ,  $\varepsilon \in N_n$ ,  $|\varepsilon| \leq n-1$ , such kind of integral has the form

$$I(f) = J(f)I(1, \dots, 1),$$

where

$$J(f) = \frac{\partial^n f(0)}{\partial \theta_1 \cdots \partial \theta_n}.$$

Since the derivative is defined with an accuracy to with an additive constant form the annihilator  ${}^\perp L_n$ ,  $L_n = \{\theta_1 \cdots \theta_n, \boldsymbol{\theta} \in \Lambda_1^n\}$ , it follows that  $J : \mathcal{P} \rightarrow \Lambda / {}^\perp L_n$  is single-valued mapping. This mapping also satisfies the conditions 1 and 2, and therefore we shall call it an integral and denote

$$J(f) = \int f(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int \theta_1 \cdots \theta_n d\theta_1 \cdots d\theta_n,$$

which has properties:

$$\begin{aligned} \int \theta_1 \cdots \theta_n d\theta_1 \cdots d\theta_n &= 1, \\ \int \frac{\partial f}{\partial \theta_j} d\theta_1 \cdots d\theta_n &= 0, \quad j = 1, \dots, n. \\ \int f(\boldsymbol{\theta}) \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_j} d\boldsymbol{\theta} &= (-1)^{1+|g|} \int \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_j} g(\boldsymbol{\theta}) d\boldsymbol{\theta}. \end{aligned} \quad (2.2)$$

In this paper, we consider functions with two ordinary even variables  $x, t$  and a Grassmann odd variable  $\theta$ . The associated space  $\mathbb{C}_\Lambda^{2,1} = \Lambda_0^2 \times \Lambda_1$  (we may take  $\Lambda_0 = \mathbb{R}$  or  $\mathbb{C}$ ) is a superspace over Grassmann algebra  $G_1(\sigma) = G_{1,0} \oplus G_{1,1}$ , whose elements have the form

$$f = f_0 + f_1\sigma.$$

where  $e = 1$  is a unit,  $\sigma$  is anticommuting generator. The monomials  $\{1, \sigma\}$  form a basis of the  $G_1(\sigma)$ ,  $\dim G_1(\sigma) = 2$ . Under traveling wave frame in space  $\mathbb{C}_\Lambda^{2,1}$ , the phase variable should have the form

$$\xi = \alpha x + \omega t + \theta \sigma.$$

Now we consider the bilinear form of the equation (1.2). By the dependent variable transformation

$$F = \partial_x \ln f(x, t, \theta), \quad (1.3)$$

where  $f(x, t, \theta) : \mathbb{C}_\Lambda^{2,1} \rightarrow \mathbb{C}_\Lambda^{1,0}$  is a superdifferential function, the equation (1.2) is then transformed into a bilinear form

$$(S_t D_t + S_t D_x^3) f(x, t, \theta) \cdot f(x, t, \theta) = 0. \quad (1.4)$$

where the Hirota bilinear differential operators  $D_x$  and  $D_t$  are defined by

$$D_x^m D_t^n f(x, t, \theta) \cdot g(x, t, \theta) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t, \theta) g(x', t', \theta')|_{x'=x, t'=t, \theta'=\theta}.$$

The super-Hirota bilinear operator is defined as [21]

$$S_t^N f(x, t, \theta) \cdot g(x, t, \theta) = \sum_{j=0}^N (-1)^{j|f| + \frac{1}{2}j(j+1)} \begin{bmatrix} N \\ j \end{bmatrix} \mathfrak{D}_t^{N-j} f(x, t, \theta) \mathfrak{D}_t^j g(x, t, \theta),$$

where the super binomial coefficients are defined by

$$\begin{bmatrix} N \\ j \end{bmatrix} = \begin{cases} \begin{bmatrix} [N/2] \\ [j/2] \end{bmatrix}, & \text{if } (N, j) \not\equiv (0, 1) \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$$

$[k]$  is the integer part of the real number  $k$  ( $[k] \leq k \leq [k] + 1$ ).

Following the Hirota bilinear theory, It is easy to find that the equation (1.2) admits one-soliton solution (also called one-supersoliton solution)

$$F_1 = \partial_x \ln(1 + e^\eta), \quad (1.5)$$

with phase variable  $\eta = kx - k^3 t + \theta \zeta + \gamma$  and  $k, \gamma \in \Lambda_0, \zeta \in \Lambda_1$ .

To apply the Hirota bilinear method for constructing periodic wave solutions of the equation (1.2), we hope to add two odd variables  $F_0, c$  and consider a more general form than the bilinear equation (1.4)

$$F = \partial_\theta^{-1} F_0 + \partial_x \ln f(x, t, \theta), \quad (2.1)$$

where  $F_0 = F_0(\theta) : \mathbb{C}_\Lambda^{2,1} \rightarrow \mathbb{C}_\Lambda^{0,1}$  is an odd special solution of the equation (1.2). Substituting (2.1) into (1.2) and integrating with respect to  $x$ , we then get the following bilinear form

$$G(S_t, D_x, D_t) f \cdot f = (S_t D_t + S_t D_x^3 + 3F_0 D_x^2 + c) f \cdot f = 0, \quad (2.2)$$

where  $c = c(\theta, t) : \mathbb{C}_\Lambda^{2,1} \rightarrow \mathbb{C}_\Lambda^{0,1}$  is an odd integration constant. For the bilinear equation (2.2), we are interested in its multi-periodic solutions in terms of the Riemann theta functions.

In the following, we introduce a super one-dimensional Riemann theta function on super space  $\mathbb{C}_\Lambda^{2,1}$  and discuss its quasi-periodicity, which plays a central role in this paper. The Riemann theta function reads

$$\vartheta(\xi, \varepsilon, s | \tau) = \sum_{n \in \mathbb{Z}} \exp[2\pi i(\xi + \varepsilon)(n + s) - \pi \tau(n + s)^2]. \quad (2.3)$$

Here the integer value  $n \in \mathbb{Z}$ ,  $s, \varepsilon \in \mathcal{C}$ , and complex phase variables  $\xi = \alpha x + \omega t + \theta \sigma + \delta$  is dependent of even variable  $x, t$  and odd  $\theta$ ; The  $\tau > 0$  is called the period matrix of the Riemann theta function. It is obvious that the Riemann theta function (2.3) converges absolutely and superdifferentiable on superspace  $\mathbb{C}_\Lambda^{2,1}$ . For the simplicity, in the case when  $s = \omega = 0$ , we denote

$$\vartheta(\xi, \tau) = \vartheta(\xi, 0, 0 | \tau).$$

**Definition 2.** A function  $f(\xi) : \mathbb{C}_\Lambda^{2,1} \rightarrow \mathbb{C}_\Lambda^{1,0}$  is said to be quasi-periodic in  $\xi = \alpha x + \omega t + \theta \sigma + \delta$  with fundamental periods  $T$ , if there exist certain constants  $a, b \in \Lambda_0$ , such that

$$f(\xi + T) = f(\xi) + a\xi + b.$$

An example of this is the ordinary Weierstrass zeta function, where

$$\zeta(\xi + \omega) = \zeta(\xi) + \eta,$$

for a fixed constant  $\eta$  when  $\omega$  is a period of the corresponding Weierstrass elliptic  $\wp$  function.

**Proposition 2.** [24] The Riemann theta function  $\vartheta(\xi, \tau)$  defined above has the periodic properties

$$\vartheta(\xi + 1 + i\tau, \tau) = \exp(-2\pi i\xi + \pi\tau)\vartheta(\xi, \tau). \quad (2.4)$$

Now we turn to see the periodicity of the solution (2.4), we take  $f(x, t, \theta)$  in the bilinear equation (2.2) as

$$f(x, t, \theta) = \vartheta(\xi, \tau),$$

where phase variable  $\xi = \alpha x + \omega t + \theta \sigma + \delta$ . By using (2.4), it is easy to see that

$$\frac{\vartheta'_\xi(\xi + i\tau, \tau)}{\vartheta(\xi + i\tau, \tau)} = -2\pi i + \frac{\vartheta'_\xi(\xi, \tau)}{\vartheta(\xi, \tau)},$$

that is,

$$\partial_\xi \ln \vartheta(\xi + i\tau, \tau) = -2\pi i + \partial_\xi \ln \vartheta(\xi, \tau). \quad (2.5)$$

According to the differential relation, we have

$$F(x, t, \theta) = F(\xi) = \partial_\theta^{-1} F_0 + \alpha \partial_\xi \ln \vartheta(\xi, \tau). \quad (2.6)$$

The equations (2.5) and (2.6) demonstrate that

$$F(\xi + 1 + i\tau) = \partial_\theta^{-1} F_0 + \alpha \partial_\xi \ln \vartheta(\xi + 1 + i\tau, \tau) = -2\pi i \alpha + F(\xi).$$

Therefore the solution  $F(\xi)$  is a quasi-periodic function with two fundamental periods 1 and  $i\tau$ .

In following, we establish uniform formula on the Riemann theta function, which will play a key role in the construction of the periodic wave solutions.

**Proposition 2.** [21] Suppose that  $f(x, t, \theta), g(x, t, \theta)$  are super differentiable on space  $\mathbb{C}_\Lambda^{2,1}$ . Then the Hirota bilinear operators  $D_x, D_t$  and super-Hirota bilinear operator  $S_x$  have properties

$$\begin{aligned} S_x^{2N} f \cdot g &= D_x^N f \cdot g, \\ D_x^m D_t^n e^{\xi_1} \cdot e^{\xi_2} &= (\alpha_1 - \alpha_2)^m (\omega_1 - \omega_2)^n e^{\xi_1 + \xi_2}, \\ S_x e^{\xi_1} \cdot e^{\xi_2} &= [\sigma_1 - \sigma_2 + \theta(\alpha_1 - \alpha_2)] e^{\xi_1 + \xi_2}, \end{aligned} \quad (2.7)$$

where  $\xi_j = \alpha_j x + \omega_j t + \theta \sigma_j + \delta_j, j = 1, 2$ . More generally, we have

$$F(S_x, D_x, D_t) e^{\xi_1} \cdot e^{\xi_2} = F(\sigma_1 - \sigma_2 + \theta(\alpha_1 - \alpha_2), \alpha_1 - \alpha_2, \omega_1 - \omega_2) e^{\xi_1 + \xi_2}, \quad (2.8)$$

where  $G(S_t, D_x, D_t)$  is a polynomial about  $S_t, D_x$  and  $D_t$ . This properties will be utilized later to explore the quasi-periodic wave solutions of the equation (1.2).

**Proposition 3.** The Hirota bilinear operators  $D_x, D_t$  and super-Hirota bilinear operator  $S_x$  have properties when they act on the Riemann theta functions

$$D_x \vartheta(\xi, \varepsilon', s' | \tau) \cdot \vartheta(\xi, \varepsilon, s | \tau) = \sum_{\mu=0,1} \partial_x \vartheta(2\xi, \varepsilon' - \varepsilon, (s' - s - \mu)/2 | 2\tau) |_{\xi=0} \vartheta(2\xi, \varepsilon' + \varepsilon, (s' + s + \mu)/2 | 2\tau), \quad (2.9)$$

$$S_t \vartheta(\xi, \varepsilon', s' | \tau) \cdot \vartheta(\xi, \varepsilon, s | \tau) = \sum_{\mu=0,1} \mathfrak{D}_t \vartheta(2\xi, \varepsilon' - \varepsilon, (s' - s - \mu)/2 | 2\tau) |_{\xi=0} \vartheta(2\xi, \varepsilon' + \varepsilon, (s' + s + \mu)/2 | 2\tau), \quad (2.10)$$

where  $\sum_{\mu=0,1}$  indicates sum with respective to  $\mu = 0, 1$ .

In general, for a polynomial operator  $G(S_t, D_x, D_t)$  about  $S_t, D_x$  and  $D_t$ , we have

$$G(S_t, D_x, D_t) \vartheta(\xi, \tau) \cdot \vartheta(\xi, \tau) = \sum_{\mu=0,1} C(\alpha, \omega, \sigma, \mu) \vartheta(2\xi, \mu/2 | 2\tau), \quad (2.11)$$

where

$$\begin{aligned} \xi &= \alpha x + \omega t + \theta \sigma + \gamma. \\ C(\alpha, \omega, \sigma, \mu | \tau) &= \sum_{n \in \mathbb{Z}} G \{ 4\pi i(n - \mu/2)\alpha, 4\pi i(n - \mu/2)\omega, \\ &\quad 4\pi i(n - \mu/2)(\sigma + \theta\omega) \} \times \exp [-2\pi\tau(n - \mu/2)^2]. \end{aligned} \quad (2.12)$$

*Proof.* By using (2.7), we have

$$\begin{aligned}
\Gamma &= S_t \vartheta(\xi, \varepsilon', s' | \tau) \cdot \vartheta(\xi, \varepsilon, s | \tau) \\
&= \sum_{m', m \in \mathbb{Z}} S_t \exp\{2\pi i(m' + s')(\xi + \varepsilon') - \pi(m' + s')^2 \tau\} \cdot \exp\{2\pi i(m + s)(\xi + \varepsilon) - \pi(m + s)^2 \tau\}, \\
&= \sum_{m', m \in \mathbb{Z}} 2\pi i(\sigma + \theta\omega)(m' - m + s' - s) \exp\{2\pi i(m' + m + s' + s)\xi - 2\pi i[(m' + s')\varepsilon' + (m + s)\varepsilon] \\
&\quad - \pi\tau[(m' + s')^2 + (m + s)^2]\} \\
&\stackrel{m=l'-m'}{=} \sum_{l', m' \in \mathbb{Z}} 2\pi i(\sigma + \theta\omega)(2m' - l' + s' - s) \exp\{2\pi i(l' + s' + s)\xi - 2\pi i[(m' + s')\varepsilon' \\
&\quad + (l' - m' + s)\varepsilon] - \pi[(m' + s')^2 + (l' - m' + s)^2]\tau\} \\
&\stackrel{l'=2l+\mu}{=} \sum_{\mu=0,1} \sum_{l, m' \in \mathbb{Z}} 2\pi i(\sigma + \theta\omega)(2m' - 2l + s' - s - \mu) \exp\{4\pi i\xi[l + (s' + s + \mu)/2] \\
&\quad - 2\pi i[(m' + s')\varepsilon' - (m' - 2l - s - \mu)\varepsilon] - \pi[(m' + s')^2 + (m' - 2l - s - \mu)^2]\tau\}
\end{aligned}$$

Let  $m' = n + l$ , and using the relations

$$\begin{aligned}
n + l + s' &= [n + (s' - s - \mu)/2] + [l + (s' + s + \mu)/2], \\
n - l - s - \mu &= [n + (s' - s - \mu)/2] - [l + (s' + s + \mu)/2],
\end{aligned}$$

we finally obtain that

$$\begin{aligned}
\Gamma &= \sum_{\mu=0,1} \left[ \sum_{n \in \mathbb{Z}} 4\pi i(\sigma + \theta\omega)[n + (s' - s - \mu)/2] \exp\{-2\pi i[n + (s' - s - \mu)/2](\varepsilon' - \varepsilon) - 2\pi\tau[n + (s' - s - \mu)/2]^2\} \right] \\
&\quad \times \left[ \sum_{l \in \mathbb{Z}} \exp\{2\pi i[l + (s' + s + \mu)/2](2\xi + \varepsilon' + \varepsilon) - 2\pi\tau[l + (s' + s + \mu)/2]^2\} \right] \\
&= \sum_{\mu=0,1} \mathfrak{D}_t \vartheta(2\xi, \varepsilon' - \varepsilon, (s' - s - \mu)/2 | 2\tau) |_{\xi=0} \vartheta(2\xi, \varepsilon' + \varepsilon, (s' + s + \mu)/2 | 2\tau).
\end{aligned}$$

In a similar way, we can prove the formulae (2.9). As a special case when  $\varepsilon = s = 0$  of the Riemann theta function (2.3), by using (2.9) and (2.10), we can prove the formula (2.11).  $\square$

From the formulae (2.11) and (2.12), it is seen that if the following equations are satisfied

$$C(\alpha, \omega, \sigma, \mu | \tau) = 0,$$

for  $\mu = 0, 1$ , then  $\vartheta(\xi, \tau)$  is a solution of the bilinear equation

$$G(S_t, D_x, D_t) \vartheta(\xi, \tau) \cdot \vartheta(\xi, \tau) = 0.$$

### 3. Quasi-periodic waves and asymptotic properties

In this section, we consider periodic wave solutions of the equation (1.2). As a simple case of the theta function (2.3) when  $N = 1, s = 0$ , we take  $f(x, t, \theta)$  as

$$f(x, t, \theta) = \vartheta(\xi, \tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi i n \xi - \pi n^2 \tau), \quad (3.1)$$

where the phase variable  $\xi = \alpha x + \omega t + \theta \sigma + \delta$ , and the parameter  $\tau > 0$ .

To let the Riemann theta function (3.1) be a solution of the bilinear equation (2.2), according to the formula (2.11), the following equations only need to be satisfied

$$\sum_{n \in \mathbb{Z}} [-16\pi^2(n - \mu/2)^2(\sigma + \theta\omega)\omega + 256\pi^4(n - \mu/2)^4(\sigma + \theta\omega)\alpha^3 - 48\pi^2(n - \mu/2)^2\alpha^2 F_0 + c] \exp[-2\pi(n - \mu/2)^2\tau] = 0, \quad \mu = 0, 1. \quad (3.2)$$

We introduce the notations by

$$\begin{aligned} \lambda &= e^{-\pi\tau/2}, \quad \vartheta_1(\xi, \lambda) = \vartheta(2\xi, 2\tau) = \sum_{n \in \mathbb{Z}} \lambda^{4n^2} \exp(4i\pi n\xi), \\ \vartheta_2(\xi, \lambda) &= \vartheta(2\xi, 0, -1/2, 2\tau) = \sum_{n \in \mathbb{Z}} \lambda^{(2n-1)^2} \exp[2i\pi(2n-1)\xi]. \end{aligned} \quad (3.3)$$

By using formula (3.3), the equation (3.2) can be written as a linear system

$$\begin{aligned} \theta \vartheta_1'' \omega^2 + (\sigma \vartheta_1'' + \alpha^3 \theta \vartheta_1^{(4)}) \omega + \vartheta_1 c + \sigma \alpha^3 \vartheta_1^{(4)} + 3\alpha^2 F_0 \vartheta_1'' &= 0, \\ \theta \vartheta_2'' \omega^2 + (\sigma \vartheta_2'' + \alpha^3 \theta \vartheta_2^{(4)}) \omega + \vartheta_2 c + \sigma \alpha^3 \vartheta_2^{(4)} + 3\alpha^2 F_0 \vartheta_2'' &= 0, \end{aligned} \quad (3.4)$$

where  $\omega \in \Lambda_0$  is even and  $c, F_0 : \mathbb{C}_{\Lambda}^{2,1} \rightarrow \mathbb{C}_{\Lambda}^{0,1}$  are odd. In addition, we have denoted derivatives of  $\vartheta_j(\xi, \lambda)$  at  $\xi = 0$  by simple notations

$$\vartheta_j^{(k)} = \vartheta_j^{(k)}(0, \lambda) = \frac{d^k \vartheta_j(\xi, \lambda)}{d\xi^k} \Big|_{\xi=0}, \quad j = 1, 2, k = 0, 1, 2, \dots$$

Moreover, these functions are independent of Grassmann variable  $\theta$  and  $\sigma$ .

We show there existence real solutions to the system (3.4). Since  $c = c(\theta, t)$  and  $F = F_0(\theta)$  are function of Grassmann variable  $\theta$ , we can expand them in the form

$$c = c_1 + c_2 \theta, \quad F_0 = f_1 + f_2 \theta, \quad (3.5)$$

where  $c_1, f_1 \in \Lambda_1$  are odd and  $c_2, f_2 \in \Lambda_0$  are even. Substituting (3.5) into (3.4) leads to

$$\begin{aligned} (\sigma \vartheta_1'' \omega + \vartheta_1 c_1 + \sigma \alpha^3 \vartheta_1^{(4)} + 3\alpha^2 \vartheta_1'' f_1) + \theta (3\alpha^2 \vartheta_1'' f_2 + \vartheta_1 c_2 + \vartheta_1'' \omega^2 + \alpha^3 \vartheta_1^{(4)} \omega) &= 0, \\ (\sigma \vartheta_2'' \omega + \vartheta_2 c_1 + \sigma \alpha^3 \vartheta_2^{(4)} + 3\alpha^2 \vartheta_2'' f_1) + \theta (3\alpha^2 \vartheta_2'' f_2 + \vartheta_2 c_2 + \vartheta_2'' \omega^2 + \alpha^3 \vartheta_2^{(4)} \omega) &= 0, \end{aligned} \quad (3.6)$$

where  $\omega, c_1, c_2, f_1$  and  $f_2$  are parameters to be determined.

Since  $\theta$  is a Grassmann variable, the system (3.6) will be satisfied provided that

$$\begin{aligned} \sigma \vartheta_1'' \omega + \vartheta_1 c_1 + \sigma \alpha^3 \vartheta_1^{(4)} + 3\alpha^2 \vartheta_1'' f_1 &= 0, \\ \sigma \vartheta_2'' \omega + \vartheta_2 c_1 + \sigma \alpha^3 \vartheta_2^{(4)} + 3\alpha^2 \vartheta_2'' f_1 &= 0 \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} 3\alpha^2 \vartheta_1'' f_2 + \vartheta_1 c_2 + \vartheta_1'' \omega^2 + \alpha^3 \vartheta_1^{(4)} \omega &= 0, \\ 3\alpha^2 \vartheta_2'' f_2 + \vartheta_2 c_2 + \vartheta_2'' \omega^2 + \alpha^3 \vartheta_2^{(4)} \omega &= 0. \end{aligned} \quad (3.8)$$

In the systems (3.7) and (3.8), it is obvious that vectors  $(\vartheta_1, \vartheta_2)^T$  and  $(\vartheta_1'', \vartheta_2'')^T$  are linear independent, and  $(\vartheta_1^{(4)}, \vartheta_2^{(4)})^T \neq 0$ . Therefore the system (3.7) admits a solution

$$\omega = -3\beta\alpha^2 + \frac{(\vartheta_2^{(4)}\vartheta_1 - \vartheta_1^{(4)}\vartheta_2)\alpha^3}{\vartheta_1''\vartheta_2 - \vartheta_2''\vartheta_1} \in \Lambda_0, \quad c_1 = \frac{(\vartheta_2^{(4)}\vartheta_1'' - \vartheta_1^{(4)}\vartheta_2'')\alpha^3\sigma}{\vartheta_1''\vartheta_2 - \vartheta_2''\vartheta_1} \in \Lambda_1, \quad (3.9)$$

here we have taken  $f_1 = \beta\sigma$ ,  $\beta \in R$  for simplicity, and other parameters  $\alpha, \tau, \sigma, \beta$  are free.

By using (3.9) and solving system (3.8), we obtain that

$$f_2 = -\beta\omega \in \Lambda_0, \quad c_2 = \frac{(\vartheta_1^{(4)}\vartheta_2'' - \vartheta_2^{(4)}\vartheta_1'')\alpha^3\omega}{\vartheta_1''\vartheta_2 - \vartheta_2''\vartheta_1} \in \Lambda_0. \quad (3.10)$$

Noting that  $\int \theta \, d\theta = 1$  and  $\int d\theta = 0$ , we have

$$\partial^{-1}F_0 = \int (\beta\sigma - \beta\omega\theta) \, d\theta = -\beta\omega.$$

In this way, we indeed can get an explicit periodic wave solution of the equation (1.12)

$$F = -\beta\omega + \partial_x \ln \vartheta(\xi, \tau), \quad (3.11)$$

with the theta function  $\vartheta(\xi, \tau)$  given by (3.1) and parameters  $\omega, c_1, c_2$  by (3.9) and (3.10), while other parameters  $\alpha, \sigma, \tau, \delta, \beta$  are free. Among them, the three parameters  $\alpha, \sigma$  and  $\tau$  completely dominate a periodic wave. In summary, the periodic wave (3.11) is real-valued and bounded for all complex variables  $(x, t, \theta)$ . It is one-dimensional, i.e. there is a single phase variable  $\xi$ , and has two fundamental periods 1 and  $i\tau$  in phase variable  $\xi$ .

In the following, we further consider asymptotic properties of the periodic wave solution. Interestingly, the relation between the one-periodic wave solution (3.11) and the one-super soliton solution (1.5) can be established as follows.

**Theorem 1.** Suppose that the  $\omega \in \Lambda_0$  and  $c \in \Lambda_1$  are given given by (3.5), (3.9) and (3.10). For the one-periodic wave solution (3.11), we let

$$\alpha = \frac{k}{2\pi i}, \quad \sigma = \frac{\zeta}{2\pi i}, \quad \delta = \frac{\gamma + \pi\tau}{2\pi i}, \quad (3.12)$$

where the  $k, \zeta$  and  $\gamma, \tau$  are the same as those in (1.5). Then we have the following asymptotic properties

$$c \longrightarrow 0, \quad \xi \longrightarrow \frac{\eta + \pi\tau}{2\pi i}, \quad \vartheta(\xi, \tau) \longrightarrow 1 + e^\eta, \quad \text{as } \lambda \rightarrow 0. \quad (3.13)$$

In other words, the periodic solution (3.11) tends to the one-soliton solution (1.5) under a small amplitude limit, that is,

$$F \longrightarrow F_1, \quad \text{as } \lambda \rightarrow 0. \quad (3.14)$$

*Proof.* Here we will directly use the system (3.4) to analyze asymptotic properties of one-periodic solution, which is more simple and effective than our original method by solving the system [16]-[20]. Since the coefficients of system (3.4) are power series about  $\lambda$ , its solution  $(\omega, c)^T$  also should be a series about  $\lambda$ .

We explicitly expand the coefficients of system (3.4) as follows

$$\begin{aligned}\vartheta_1(0, \lambda) &= 1 + 2\lambda^4 + \dots, \quad \vartheta_1''(0, \lambda) = -32\pi^2\lambda^4 + \dots, \\ \vartheta_1^{(4)}(0, \lambda) &= 512\pi^4\lambda^4 + \dots, \quad \vartheta_2(0, \lambda) = 2 + 2\lambda^8 + \dots \\ \vartheta_2''(0, \lambda) &= -8\pi^2 - 72\pi^2\lambda^8 + \dots, \quad \vartheta_2^{(4)}(0, \lambda) = 32\pi^4 + 2592\pi^4\lambda^8 + \dots.\end{aligned}\tag{3.15}$$

Let the solution of the system (3.4) be in the form

$$\begin{aligned}\omega &= \omega_0 + \omega_1\lambda + \omega_2\lambda^2 + \dots = \omega_0 + o(\lambda), \\ c &= b_0 + b_1\lambda + b_2\lambda^2 + \dots = b_0 + o(\lambda),\end{aligned}\tag{3.16}$$

where  $\omega_j \in \Lambda_0$ ,  $b_j \in \Lambda_1$ ,  $j = 0, 1, 2, \dots$

Substituting the expansions (3.11) and (3.12) into the system (3.5) and letting  $\lambda \rightarrow 0$ , we immediately obtain the following relations

$$b_0 = 0, \quad -8\pi^2\sigma\omega_0 + 2b_0 + 32\pi^4\sigma\alpha^3 = 0,$$

which has a solution

$$b_0 = 0, \quad w_0 = 4\pi^2\alpha^3.$$

Then from the relations (3.12) and (3.16), we have

$$c \rightarrow 0, \quad 2\pi i\omega \rightarrow 8\pi^3 i\alpha^3 = -k^3, \quad \text{as } \lambda \rightarrow 0,$$

and thus

$$\begin{aligned}\hat{\xi} &= 2\pi i\xi - \pi\tau = kx + 2\pi i\omega t + \theta\zeta + \gamma \\ &\rightarrow kx - k^3t + \theta\zeta + \gamma = \eta, \quad \text{as } \lambda \rightarrow 0,\end{aligned}\tag{3.17}$$

It remains to show that the one-periodic wave (3.11) possesses the same form with the one-soliton solution (1.5) under the limit  $\lambda \rightarrow 0$ . For this purpose, we first expand the Riemann theta function  $\vartheta(\xi, \tau)$  in the form

$$\vartheta(\xi, \tau) = 1 + \lambda^2(e^{2\pi i\xi} + e^{-2\pi i\xi}) + \lambda^8(e^{4\pi i\xi} + e^{-4\pi i\xi}) + \dots.$$

By using the (3.12) and (3.17), it follows that

$$\begin{aligned}\vartheta(\xi, \tau) &= 1 + e^{\hat{\xi}} + \lambda^4(e^{-\hat{\xi}} + e^{2\hat{\xi}}) + \lambda^{12}(e^{-2\hat{\xi}} + e^{3\hat{\xi}}) + \dots \\ &\rightarrow 1 + e^{\hat{\xi}} \rightarrow 1 + e^{\eta}, \quad \text{as } \lambda \rightarrow 0,\end{aligned}$$

which implies (3.13) and (3.14). Therefore we conclude that the one-periodic solution (3.11) just goes to the one-soliton solution (1.5) as the amplitude  $\lambda \rightarrow 0$ .  $\square$

## 4. Discussion on the conditions of $N$ -periodic wave solutions

In this section, we consider condition for  $N$ -periodic wave solutions of the equation (1.2). The theta function is taken the form

$$\vartheta(\boldsymbol{\xi}, \boldsymbol{\tau}) = \vartheta(\xi_1, \dots, \xi_N, \tau) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \exp\{2\pi i \langle \boldsymbol{\xi}, \mathbf{n} \rangle - \pi \langle \boldsymbol{\tau}, \mathbf{n} \rangle\}, \quad (4.1)$$

where  $\mathbf{n} = (n_1, \dots, n_N)^T \in \mathbb{Z}^N$ ,  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)^T \in \mathcal{C}^N$ ,  $\xi_i = \alpha_j x + \omega_j t + \theta \sigma_j + \delta_j$ ,  $j = 1, \dots, N$ ,  $\tau$  is a  $N \times N$  symmetric positive definite matrix.

To make the theta function (4.1) satisfy the bilinear equation (2.2), we obtain that according to the formula (2.11)

$$\sum_{\mu=0,1} \sum_{n_1, \dots, n_N=-\infty}^{\infty} G \left\{ 4\pi i \sum_{j=1}^N (n_j - \mu_j/2) \alpha_j, 4\pi i \sum_{j=1}^N (n_j - \mu_j/2) \omega_j, \right. \\ \left. 4\pi i \sum_{j=1}^N (n_j - \mu_j/2) (\sigma_j + \theta \omega_j) \right\} \times \exp \left[ -2\pi \sum_{j,k=1}^N (n_j - \mu_j/2) \tau_{jk} (n_k - \mu_k/2) \right] = 0. \quad (4.2)$$

Now we consider the number of equation and some unknown parameters. Obviously, in the case of supersymmetric equations, the number of constraint equations of the type (4.2) is  $2^{N+1}$ , which is two times of the constraint equations needed in the case of ordinary equations [16]-[20]. On the other hand we have parameters  $\tau_{ij} = \tau_{ji}, c_1, c_2, f_1, f_2, \alpha_i, \omega_i$ , whose total number is  $\frac{1}{2}N(N+1) + 2N + 4$ . Among them,  $2N$  parameters  $\tau_{ii}, \omega_i$  are taken to be the given parameters related to the amplitudes and wave numbers (or frequencies) of  $N$ -periodic waves;  $\frac{1}{2}N(N+1)$  parameters  $\tau_{ij}$  implicitly appear in series form, which is general can not to be solved explicit. Hence, the number of the explicit unknown parameters is only  $N+4$ . The number of equations is larger than the unknown parameters in the case when  $N \geq 2$ . In this paper, we consider one-periodic wave solution of the equation (1.2), which belongs to the cases when  $N = 1$ . There are still certain difficulties in the calculation for the case  $N \geq 2$ , which will be considered in our future work.

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